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# On a class of degenerate elliptic operators in $L^1$ spaces with respect to invariant measures

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**Abstract** We consider a class of second order degenerate elliptic operators arising from second order stochastic differential equations in $\mathbb{R}^n$ perturbed by noise. We study realizations of such operators in $L^1$ spaces with respect to their (explicit) invariant measure, proving that they are $m$-dissipative.

**Keywords** Second order stochastic equations · Degenerate elliptic equations · Kolmogorov equations · Invariant measures

**Mathematics Subject Classification** 35J70 · 60H10 · 37L40

## 1 Introduction

We consider a class of Kolmogorov operators associated to second order stochastic differential equations in $\mathbb{R}^n$, such as

$$\begin{cases} y''(t) = -My(t) - y'(t) - DU(y(t)) + W'(t), \\ y(0) = y_0, \quad y'(0) = x_0, \end{cases} \tag{1.1}$$

where $M$ is a symmetric positive definite matrix, $U \in C^1(\mathbb{R}^n, \mathbb{R})$ and $W$ is a standard Brownian motion in $\mathbb{R}^n$.

Equation (1.1) is a model for the motion $y(t)$ of a particle of mass 1 subject to a force field $-My - DU(y)$ perturbed by noise. The term $-y'(t)$ describes the friction

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to the motion, proportional to the velocity. See e.g. [2] and the references therein. Setting $y'(t) = x(t)$ the differential equation in (1.1) is rewritten as a system,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = B\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \begin{pmatrix} DU(y(t)) \\ 0 \end{pmatrix} + \begin{pmatrix} W'(t) \\ 0 \end{pmatrix}, \tag{1.2}$$

where

$$B = \begin{pmatrix} -\mathbb{I} & -M \\ \mathbb{I} & 0 \end{pmatrix}, \tag{1.3}$$

and $\mathbb{I}$ is the identity matrix in $\mathbb{R}^n$.

It is well known that if $U$ is sufficiently smooth and it has bounded second order derivatives, problem (1.1) has a unique strong solution $(x(t), y(t))$.

The corresponding Kolmogorov operator in $\mathbb{R}^{2n}$ is given by

$$(K\varphi)(x,y) = \frac{1}{2}\,\Delta_x\varphi - \langle My + D_yU(y) + x, D_x\varphi\rangle + \langle x, D_y\varphi\rangle, \tag{1.4}$$

and its formal adjoint $K^*$ is given by

$$(K^*\rho)(x,y) = \frac{1}{2}\,\Delta_x\rho - \langle x, D_y\rho\rangle + \langle My + D_yU(y) + x, D_x\rho\rangle + n\rho. \tag{1.5}$$

It is easy to see that the function $\rho(x,y)$, defined by

$$\rho(x,y) = \mathrm{e}^{-(\langle My,y\rangle + |x|^2)}\mathrm{e}^{-2U(y)}$$

satisfies $K^*\rho = 0$, and that

$$Z := \int_{\mathbb{R}^{2n}} \rho(x,y)\mathrm{d}x\,\mathrm{d}y < +\infty, \tag{1.6}$$

provided that $U$ is bounded from below.

In fact, under suitable assumptions it follows from [4] that problem (1.2) possesses a unique probability invariant measure $\mu$ given by $\mu(\mathrm{d}x, \mathrm{d}y) = Z^{-1}\rho(x,y)\mathrm{d}x\mathrm{d}y$.

Therefore it is of interest to study the realization of the Kolmogorov operator $K$, defined in a suitable domain (see Sect. 3 below), in the spaces $L^p(\mathbb{R}^{2n}, \mu)$, under minimal assumptions on $U$.

The simplest situation is $n = 1$, $M = 1$, $U \equiv 0$, in which case $\mu$ is the probability Gaussian measure in $\mathbb{R}^2$ and $K$ is the Kolmogorov operator

$$(K\varphi)(x,y) = \frac{1}{2}\,\varphi_{xx} + x\varphi_y - (y+x)\varphi_x.$$

For general dimension $n$ and for general matrix $M$, if $U \equiv 0$ then $K$ is still a hypoelliptic Kolmogorov operator, because it can be written as

$$(K\varphi)(z) = \frac{1}{2}\operatorname{Tr} QD^2\varphi(z) + \langle Bz, D\varphi(z)\rangle$$

where the matrices

$$Q = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -\mathbb{I} & -M \\ \mathbb{I} & 0 \end{pmatrix}$$

satisfy the controllability or hypoellipticity condition

$$\operatorname{Rank}(Q^{1/2}, BQ^{1/2}) = 2n.$$

Integration by parts shows that for each regular $u$, $v$ with good behavior at infinity (say for instance $u, v \in C_0^\infty(\mathbb{R}^{2n})$) we have

$$\int_{\mathbb{R}^{2n}} Ku\, v\, \mathrm{d}\mu = -\frac{1}{2}\int_{\mathbb{R}^{2n}} \langle D_x u, D_x v\rangle \mathrm{d}\mu + \int_{\mathbb{R}^{2n}} (\langle D_y u, D_x v\rangle - \langle D_x u, D_y v\rangle)\mathrm{d}\mu, \tag{1.7}$$

where $D_x$, $D_y$ denote the gradients with respect to the variables $x$, $y$ only. Therefore $K$ is not symmetric in $L^2(\mathbb{R}^{2n}, \mu)$. It is naturally associated to the quadratic form

$$a(u,v) = \frac{1}{2}\int_{\mathbb{R}^{2n}} \langle D_x u, D_x v\rangle \mathrm{d}\mu - \int_{\mathbb{R}^{2n}} (\langle D_y u, D_x v\rangle - \langle D_x u, D_y v\rangle)\mathrm{d}\mu,$$

which is well defined and continuous in $H^1(\mathbb{R}^{2n}, \mu)$. This is the space of the functions $u \in L^2(\mathbb{R}^{2n}, \mu) \cap H^1_{loc}(\mathbb{R}^{2n})$ whose first order derivatives belong to $L^2(\mathbb{R}^{2n}, \mu)$, and it is a Hilbert space with its natural scalar product

$$\langle u, v\rangle_{H^1(\mathbb{R}^{2n},\mu)} := \int_{\mathbb{R}^{2n}} u\, v\, \mathrm{d}\mu + \int_{\mathbb{R}^{2n}} \langle Du, Dv\rangle \mathrm{d}\mu.$$

We have

$$a(u,u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |D_x u|^2 \mathrm{d}\mu, \quad u \in H^1(\mathbb{R}^{2n}, \mu),$$

therefore for any $\lambda > 0$ the form $(u, v) \mapsto a(u, v) + \lambda\langle u, v\rangle_{L^2}$ is not coercive, and the Lax-Milgram lemma cannot be used to find solutions to

$$\lambda\langle u, v\rangle_{L^2} + a(u,v) = \langle f, v\rangle_{L^2}, \quad v \in H^1(\mathbb{R}^{2n}, \mu), \tag{1.8}$$

i.e. to find weak solutions to $\lambda u - Ku = f$ for $f \in L^2(\mathbb{R}^{2n}, \mu)$. However, taking $v = u$ and $v = sign\ u$ (or a smooth approximation of $sign\ u$) in (1.7), it follows in a more or less standard way that the realization of $K$ with domain $C_0^\infty(\mathbb{R}^{2n})$ (or another space of good enough functions) is dissipative in $L^2(\mathbb{R}^{2n}, \mu)$ and in $L^1(\mathbb{R}^{2n}, \mu)$. By interpolation, it is dissipative in $L^p(\mathbb{R}^{2n}, \mu)$ for each $p \in [1, 2]$, and hence it is closable with dissipative closure $K_p$.

In this paper we focus our attention on the operator $K_1$. We show that if

$$\int_{\mathbb{R}^n} (|U(y)|^2 + |DU(y)|^2)\mathrm{e}^{-2U(y)}\mathrm{d}y < +\infty, \tag{1.9}$$

then $K_1$ is $m$-dissipative in $L^1(\mathbb{R}^{2n}, \mu)$, i.e. it is dissipative and the range of $\lambda I - K_1$ is the whole $L^1(\mathbb{R}^{2n}, \mu)$ for each $\lambda > 0$.

By the Lumer–Phillips Theorem it suffices to prove essential $m$-dissipativity. This is done by perturbation, using existence, uniqueness, and regularity results for the case $U \equiv 0$. In this case Assumption (1.6) is satisfied because $M$ is positive definite.

Every $m$-dissipative operator with dense domain is the infinitesimal generator of a strongly continuous contraction semigroup. Denoting by $T(t)$ the semigroup generated by $K_1$, we prove that

$$\int_{\mathbb{R}^{2n}} T(t)f\, \mathrm{d}\mu = \int_{\mathbb{R}^{2n}} f\, \mathrm{d}\mu$$

for each $t > 0$ and $f \in L^1(\mathbb{R}^{2n}, \mu)$, i.e. $\mu$ is an invariant measure for $T(t)$.

We notice that assumption (1.9) does not guarantee existence in the large of the strong solution to (1.1). If all the solutions to (1.1) exist in the large, for each Borel measurable and bounded function $f$ it is natural to have the equality

$$(T(t)f)(x_0, y_0) = \mathbb{E}[f(x(t, x_0, y_0), y(t, x_0, y_0))], \quad t > 0, \tag{1.10}$$

where $(x(t, x_0, y_0), y(t, x_0, y_0))$ is the solution to (1.2) with initial data $x(0, x_0, y_0) = x_0$, $y(0, x_0, y_0) = y_0$.

But in general it is not clear whether the solutions to (1.2) exists in the large or not, while the left hand side of (1.10) still makes sense and it can be used to solve in a weak sense (1.2) using the theory of Dirichlet forms. See [7], [3], [8], [10].

For a detailed treatment of nondegenerate second order elliptic operators in $L^1$ spaces with respect to invariant measures we refer to [9]. To our knowledge, this is the first study in the degenerate case.

## 2 The case $U \equiv 0$

We need a preliminary study of the realization of $K$ in the space $C_b(\mathbb{R}^{2n})$ of the continuous and bounded functions from $\mathbb{R}^{2n}$ to $\mathbb{R}$, in the case $U \equiv 0$. To this aim we use some results of [6], to which we refer for more details, proofs and comments.

It is well known that the solution of (1.1) is given by the Ornstein-Uhlenbeck process

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tB} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_0^t e^{(t-s)B} \begin{pmatrix} dW(s) \\ 0 \end{pmatrix}.$$

The corresponding transition semigroup—the Ornstein-Uhlenbeck semigroup—is defined by

$$(R(t)\varphi)(z) = \frac{1}{(2\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^{2n}} e^{-\langle Q_t^{-1}\xi, \xi\rangle/2} \varphi(e^{tB}z - \xi) d\xi, \quad t > 0,$$
$$z = (x, y) \in \mathbb{R}^{2n}, \tag{2.1}$$

where $Q_t$ is the matrix

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \quad 0 \le t \le +\infty. \tag{2.2}$$

Note that the eigenvalues of $B$ have negative real part, so that $e^{sB}$, $e^{sB^*}$ decay exponentially as $s \to +\infty$ and $Q_t$ is well defined at $t = +\infty$ too.

Setting for $\operatorname{Re}\lambda > 0$

$$(F(\lambda)\varphi)(z) = \int_0^{+\infty} e^{-\lambda t}(R(t)\varphi)(z) dt, \quad z \in \mathbb{R}^{2n}, \ \varphi \in C_b(\mathbb{R}^{2n}),$$

then $F(\lambda)$ is one to one in $C_b(\mathbb{R}^{2n})$ and it satisfies the resolvent identity in the half–plane $\operatorname{Re}\lambda > 0$. Consequently there exists a unique closed operator $L : D(L) \mapsto C_b(\mathbb{R}^{2n})$

such that

$$F(\lambda) = (\lambda I - L)^{-1}, \quad \mathrm{Re}\,\lambda > 0.$$

$L$ is called the *generator* of $R(t)$. Note that $R(t)$ is not strongly continuous in $C_b(\mathbb{R}^{2n})$ and even in $BUC(\mathbb{R}^{2n})$, see [6, Proposition 6.3], so that it has no infinitesimal generator in the usual sense.

$L$ is a realization in $C_b(\mathbb{R}^{2n})$ of the Kolmogorov (or degenerate Ornstein–Uhlenbeck) operator $\mathcal{L}$ defined by

$$(\mathcal{L}u)(x,y) = \frac{1}{2}\,\Delta_x u(x,y) - \langle My + x, D_x u(x,y)\rangle + \langle x, D_y u(x,y)\rangle. \tag{2.3}$$

**Proposition 2.1** *Let $u \in D(L)$, i.e. $u = R(\lambda, L)f$ for some $\lambda > 0$ and $f \in C_b(\mathbb{R}^{2n})$. Then $D_{x_i}u \in C_b(\mathbb{R}^{2n})$ for $i = 1, \dots, n$, and there is $C > 0$, independent of $u$ and $f$, such that*

$$\|D_{x_i}u\|_\infty \le \frac{C}{\sqrt{\lambda}}\|f\|_\infty. \tag{2.4}$$

*If in addition $f \in C^1_b(\mathbb{R}^{2n})$,*(1) *then $u$, $D_{x_i}u \in C^1_b(\mathbb{R}^{2n})$ for $i = 1, \dots, n$, and there is $C > 0$, independent of $u$ and $f$, such that*

$$\|\,|Du|\,\|_\infty \le \frac{C}{\lambda}\|\,|Df|\,\|_\infty, \quad \|D_{x_ix_j}u\|_\infty + \|D_{x_iy_j}u\|_\infty \le \frac{C}{\sqrt{\lambda}}\|\,|Df|\,\|_\infty. \tag{2.5}$$

*Moreover, $Lu$ is given by the right hand side of* (2.3).

*Proof* We use the following estimate from [6, Proposition 3.2],

$$|(D_xR(t)f)(z)| \le ct^{-1/2}\,\|f\|_\infty, \quad f \in C_b(\mathbb{R}^{2n}),\ \ t > 0,\ \ z \in \mathbb{R}^{2n}. \tag{2.6}$$

Replacing in the formula for the resolvent, (2.4) follows.

Concerning (2.5), since $DR(t)f = \mathrm{e}^{tB^*}R(t)Df$ for each $f \in C^1_b(\mathbb{R}^{2n})$ and $t > 0$, then $D_{x_i}R(t)f \in C^1_b(\mathbb{R}^{2n})$ for $f \in C^1_b(\mathbb{R}^{2n})$, $i = 1, \dots, n$, and $D(D_{x_i}R(t)\varphi) = D_{x_i}(DR(t)\varphi) = D_{x_i}\mathrm{e}^{tB^*}R(t)Df$, so that

$$\|D_{x_i}R(t)f\|_{C^1_b(\mathbb{R}^{2n})} \le ct^{-1/2}\,\|f\|_{C^1_b(\mathbb{R}^{2n})}, \quad f \in C^1_b(\mathbb{R}^{2n}),\ \ t > 0,\ \ i = 1, \dots, n. \tag{2.7}$$

Replacing again in the formula for the resolvent, (2.5) follows. From [6, Theorem 6.2] we know that $u$ is a distributional solution to $\lambda u - \mathcal{L}u = f$, in the sense of the tempered distributions. Since $u$ is regular enough, it is a classical solution. □

In the following we shall need that $Lu$ is nonpositive at each maximum point and nonnegative at each minimum point, which is obvious for smooth functions $u$ but it is not immediate for general $u \in D(L)$.

**Lemma 2.2** *Let $u \in D(L)$. At any local maximum (resp. minimum) point $z_0$ for $u$ we have $Lu(z_0) \le 0$ (respectively $Lu(z_0) \ge 0$).*

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1 $C^1_b(\mathbb{R}^{2n})$ is the space of all bounded and continuously differentiable functions from $\mathbb{R}^{2n}$ to $\mathbb{R}$ with bounded derivatives, endowed with the norm $\|f\|_{C^1_b(\mathbb{R}^{2n})} = \|f\|_\infty + \sum_{i=1}^{2n}\|D_if\|_\infty$.

*Proof* The statement is obvious if $u \in C^1(\mathbb{R}^{2n})$ and it has continuous derivatives $D_{x_ix_i}u$, $i = 1, \ldots, n$. For general $u \in D(L)$, let $z_0$ be a local maximum point for $u$. Let $r > 0$ be such that $u(z_0) \geq u(z)$ for $|z - z_0| \leq r$. Possibly adding a constant, we may assume that $u(z_0) > 0$ and $u(z) \geq 0$ for each $z \in B(z_0, r)$. Let $\theta$ be a smooth cutoff function, such that $0 \leq \theta(z) < 1$ for each $z \neq z_0$, $\theta(z_0) = 1$, $\theta \equiv 0$ outside $B(z_0, r)$, and all the second order derivatives of $\theta$ vanish at $z_0$. Then the function

$$\widetilde{u}(z) := u(z)\theta(z), \quad z \in \mathbb{R}^{2n},$$

has $z_0$ as its unique maximum point, it belongs to $D(L)$ and

$$L\widetilde{u} = Lu \cdot \theta + L\theta \cdot u + \langle D_x u, D_x\theta\rangle. \tag{2.8}$$

This can be seen as follows. Fix $\lambda > 0$ and set $\lambda u - Lu = f$, then approach $f$ by a sequence $(f_k)$ of uniformly bounded functions belonging to $C^1_b(\mathbb{R}^{2n})$ that converge to $f$ uniformly on each compact subset of $\mathbb{R}^{2n}$ (for instance, we may approximate $f$ by convolution with smooth mollifiers). Then $R(t)f_k$ converges to $R(t)f$ uniformly on each compact subset of $\mathbb{R}^{2n}$, and by dominated convergence $u_k := R(\lambda, L)f_k$ converges pointwise to $u = R(\lambda, L)f$ and $D_{x_i}u_k$ converges pointwise to $D_{x_i}u$ for $i = 1, \ldots, n$. By difference, $Lu_k$ converges pointwise to $Lu$. Since $L(u_k\theta) = Lu_k \cdot \theta + L\theta \cdot u_k + \langle D_x u_k, D_x\theta\rangle$ by Proposition 2.1, letting $k \to +\infty$ we get (2.8).

Now we go on as in the proof of [5, Proposition 3.1.10]. Since $\lambda\widetilde{u} - L\widetilde{u} := \varphi$ is uniformly continuous and bounded, there is a sequence $(\varphi_k)$ of uniformly bounded functions belonging to $C^1_b(\mathbb{R}^{2n})$ that converge to $\varphi$ uniformly on $\mathbb{R}^{2n}$. The functions $\widetilde{u}_k := R(\lambda, L)\varphi_k$ belong to $C^1_b(\mathbb{R}^{2n})$ and they are twice continuously differentiable with respect to the $x$ variables. They converge uniformly to $u = R(\lambda, L)\varphi$, and by difference $L\widetilde{u}_k$ converges uniformly to $L\widetilde{u}$.

Since $\widetilde{u}_k$ converges to $\widetilde{u}$ in $L^\infty$ and $\widetilde{u} \equiv 0$ outside $B(z_0, r)$, also $\widetilde{u}_k$ has maximum points for $k$ large enough, and since $z_0$ is the unique maximum point of $\widetilde{u}$, there is a sequence $(z_k)$ of maximum points for $\widetilde{u}_k$ that goes to $z_0$ as $k \to +\infty$. Since $L\widetilde{u}_k(z_k) \leq 0$, then also $L\widetilde{u}(z_0) \leq 0$. But $L\widetilde{u}(z_0) = Lu(z_0)$, and the statement follows. □

## 3 The general case

Here we assume that (1.9) holds. It follows that

$$\int_{\mathbb{R}^{2n}} (|U(y)|^2 + |DU(y)|^2)\mathrm{d}\mu < +\infty, \tag{3.1}$$

i.e. $(x, y) \mapsto U(y) \in H^1(\mathbb{R}^{2n}, \mu)$. We define the operator $K$ on the domain of $L$ by

$$(Ku)(x, y) = (Lu)(x, y) - \langle DU(y), D_x u(x, y)\rangle, \quad u \in D(L). \tag{3.2}$$

Note that $\langle DU, D_x u\rangle \in L^1(\mathbb{R}^{2n}, \mu)$ for each $u \in D(L)$ because $D_x u$ is bounded by Proposition 2.1 and $|DU| \in L^2(\mathbb{R}^{2n}, \mu) \subset L^1(\mathbb{R}^{2n}, \mu)$.

Let us prove two basic identities.

**Proposition 3.1** *For any $u \in D(L)$ we have*

$$\int_{\mathbb{R}^{2n}} Ku\,u\,\mathrm{d}\mu = -\frac{1}{2}\int_{\mathbb{R}^{2n}} |D_x u|^2 \mathrm{d}\mu. \tag{3.3}$$

*More generally, if $g \in C^1(\mathbb{R})$ has bounded derivative $g'$, then*

$$\int_{\mathbb{R}^{2n}} Ku\,g(u)\mathrm{d}\mu = -\frac{1}{2}\int_{\mathbb{R}^{2n}} |D_x u|^2 g'(u)\mathrm{d}\mu. \tag{3.4}$$

*Proof* Both (3.3) and (3.4) follow just integrating by parts if $u \in D(L) \cap C_b^1(\mathbb{R}^{2n})$ and also the second order derivatives $D_{x_i y_i} u$ are continuous and bounded. If not, we set $\lambda u - Lu = f$, and we approach $f$ by a sequence $(f_k)$ of uniformly bounded functions belonging to $C_b^1(\mathbb{R}^{2n})$ that converge to $f$ uniformly on each compact subset of $\mathbb{R}^{2n}$, as in the proof of Lemma 2.2. The functions $u_k := R(\lambda, L)f_k$, $D_{x_i} u_k$ and $Lu_k$ converge pointwise to $u$, to $D_{x_i} u$, and to $Lu$, respectively, with dominated convergence. Moreover each $u_k$ is in $D(L) \cap C_b^1(\mathbb{R}^{2n})$ and its second order derivatives $D_{x_i y_i} u_k$ are continuous and bounded. Therefore, (3.3) and (3.4) are true with $u_k$ replacing $u$. Letting $k \to \infty$ we obtain (3.3) and (3.4) for $u$. □

Using (3.4) we can prove that $K$ is dissipative in $L^1(\mathbb{R}^{2n}, \mu)$.

**Proposition 3.2** *For each $u \in D(L)$ and $\lambda > 0$ we have*

$$\|u\|_{L^1(\mathbb{R}^{2n},\mu)} \le \frac{1}{\lambda}\|\lambda u - Lu\|_{L^1(\mathbb{R}^{2n},\mu)}.$$

*Proof* Set $\lambda u - Ku = f$ and

$$g_k(w) = \frac{2}{\pi}\arctan(kw), \quad k \in \mathbb{N},\ w \in \mathbb{R}.$$

The sequence $(g_k(u(z)))$ approaches *sign* $u(z)$, as $k \to \infty$, at each $z$, and for each $k$ we have

$$\int_{\mathbb{R}^{2n}} \lambda u\,g_k(u)\mathrm{d}\mu - \int_{\mathbb{R}^{2n}} Ku\,g_k(u)\mathrm{d}\mu = \int_{\mathbb{R}^{2n}} f\,g_k(u)\mathrm{d}\mu.$$

By formula (3.4) we get

$$\int_{\mathbb{R}^{2n}} Ku\,g_k(u)\mathrm{d}\mu = -\frac{1}{2}\int_{\mathbb{R}^{2n}} |D_x u|^2 g_k'(u)\mathrm{d}\mu \le 0,$$

so that

$$\int_{\mathbb{R}^{2n}} \lambda u\,g_k(u)\mathrm{d}\mu \le \int_{\mathbb{R}^{2n}} f\,g_k(u)\mathrm{d}\mu \le \int_{\mathbb{R}^{2n}} |f|\mathrm{d}\mu$$

and letting $k \to +\infty$ we obtain $\lambda\|u\|_{L^1(\mathbb{R}^{2n},\mu)} \le \|f\|_{L^1(\mathbb{R}^{2n},\mu)}$ by dominated convergence. □

Now we need some *a priori* estimates in the sup norm for the solution to $\lambda u - Lu - \langle F, D_x u\rangle = f$ with a vector field $F \in C_b(\mathbb{R}^{2n}, \mathbb{R}^n)$. They yield dissipativity of $L + \langle F, D_x\cdot\rangle$ in $C_b(\mathbb{R}^{2n})$, and they will be obtained extending the Maximum Principle to our situation.

**Lemma 3.3** *Let $F \in C_b(\mathbb{R}^{2n}, \mathbb{R}^n)$ and let $u \in D(L)$. For $\lambda > 0$ set $\lambda u - Lu - \langle F, D_x u\rangle = f$. Then*

$$\|u\|_\infty \le \frac{1}{\lambda}\|f\|_\infty.$$

*If in addition $f(z) \ge 0$ for each $z \in \mathbb{R}^{2n}$, then $u(z) \ge 0$ for each $z \in \mathbb{R}^{2n}$.*

*Proof* Once Lemma 2.2 is established the proof is similar to the one in the case of nondegenerate principal part; we write it below for the reader's convenience.

If there is $z_0 \in \mathbb{R}^{2n}$ such that $\|u\|_\infty = \pm u(z_0)$ the first statement follows immediately from Lemma 2.2. If not, we may assume without loss of generality that $\|u\|_\infty = \sup u$ and that there is a sequence $(z_k)$ of points in $\mathbb{R}^{2n}$ such that $|z_k| \to +\infty$, $u(z_k) \ge \|u\|_\infty - 1/k$.

Let $\theta$ be a smooth cutoff function, such that $0 \le \theta(z) \le 1$ for every $z$, $\theta \equiv 1$ on $B(0,1)$, $\theta \equiv 0$ outside $B(0,2)$. Set $\theta_k(z) = \theta((z - z_k)/|z_k|)$, so that

$$\sup_{k\in\mathbb{N}} \|L\theta_k + \langle F, D_x\theta_k\rangle\|_\infty < \infty,$$

and set

$$u_k(z) = u(z) + \frac{2}{k}\,\theta_k(z), \quad z \in \mathbb{R}^{2n}.$$

Then $u_k$ converges to $u$ uniformly, and for every $k$, $\sup u_k = \max u_k$. Let $w_k \in \mathbb{R}^{2n}$ be any maximum point for $u_k$. Then $u_k(w_k)$ goes to $\|u\|_\infty$ as $k \to +\infty$. Moreover,

$$\begin{aligned}&\lambda u_k(w_k) - Lu_k(w_k) - \langle F(w_k), D_x u_k(w_k)\rangle \\ &\quad = f(w_k) + \frac{2}{k}\left(\lambda\theta_k(w_k) - L\theta_k(w_k) - \langle F(w_k), D_x\theta_k(w_k)\rangle\right),\end{aligned} \tag{3.5}$$

so that

$$u_k(w_k) \le \frac{1}{\lambda}\,\|f\|_\infty + \frac{2}{k\lambda}\,\|\lambda\theta_k - L\theta_k - \langle F, D_x\theta_k\rangle\|_\infty.$$

Letting $k \to \infty$ we get the first statement.

To prove the second statement we follow the same procedure. It is enough (in fact, it is equivalent) to show that if $f(z) \le 0$ for each $z$, then $u(z) \le 0$ for each $z$. Assume by contradiction that $f(z) \le 0$ for each $z$ but $u(z) > 0$ at some $z$. Then $\sup u > 0$. If $\sup u = \max u$ and $z_0$ is a maximum point then $\lambda u(z_0) - Lu(z_0) > 0$, a contradiction. If $u$ has not maximum points, define $w_k$, $u_k$ as above. Since $f(w_k) \le 0$, formula (3.5) implies

$$u_k(w_k) \le \frac{2}{k\lambda}\,\|\lambda\theta_k - L\theta_k - \langle F, D_x\theta_k\rangle\|_\infty$$

that converges to 0 as $k \to +\infty$. Therefore, $\sup u = \lim_{k\to+\infty} \max u_k \le 0$, and the second statement follows. □

Since $U \in H^1(\mathbb{R}^{2n}, \mu)$, it may be approximated by a sequence of $C_0^\infty(\mathbb{R}^{2n})$ functions, as the next lemma shows.

**Lemma 3.4** *There is a sequence of $C_0^\infty(\mathbb{R}^{2n})$ functions $U_k = U_k(y)$ such that*

$$\lim_{k\to\infty} \|U_k - U\|_{H^1(\mathbb{R}^{2n},\mu)} = 0.$$

*Proof* Let $\nu$ be the measure in $\mathbb{R}^n$ defined by $\nu(\mathrm{d}y) = \exp(-2U(y) - \langle My, y\rangle)\mathrm{d}y$. From [1, Lemma 2.2] we obtain that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n, \nu)$, and therefore there is a sequence of functions $U_k$ in $C_0^\infty(\mathbb{R}^n)$ that approach $U$ in $H^1(\mathbb{R}^n, \nu)$. The functions $(x, y) \mapsto U_k(y)$ belong to $C_0^\infty(\mathbb{R}^{2n})$ and

$$\|U_k - U\|_{H^1(\mathbb{R}^{2n},\mu)} \le \frac{\pi^{n/2}}{Z}\|U_k - U\|_{H^1(\mathbb{R}^n,\nu)},$$

where $Z$ is defined by (1.6), and the statement follows. □

**Proposition 3.5** *Let $U_k$ be the functions given by Lemma 3.4. For any $\lambda > 0$ and $f \in C_b(\mathbb{R}^{2n})$ the equation*

$$\lambda u_k - Lu_k + \langle DU_k, D_x u_k\rangle = f, \tag{3.6}$$

*has a unique solution $u_k \in D(L)$.*

*Moreover, there is $\lambda_k > 0$ such that if $\lambda > \lambda_k$ and $f \in C_b^1(\mathbb{R}^{2n})$ then $D_{x_i}u_k \in C_b^1(\mathbb{R}^{2n})$ for every $i = 1, \ldots, n$.*

*Proof* Setting $\lambda u_k - Lu_k = \varphi$, Eq. (3.6) is equivalent to

$$\varphi = -\langle DU_k, D_x R(\lambda, L)\varphi\rangle + f \tag{3.7}$$

Using Corollary 2.1, we see that the operator $\varphi \mapsto -\langle DU_k, D_x R(\lambda, L)\varphi\rangle$ is a contraction in $C_b(\mathbb{R}^{2n})$ if $\lambda$ is large enough, say $\lambda > \lambda_0$. In this case (3.6) has a unique solution $u_k$ in $D(L)$, $u_k = R(\lambda, L)\varphi$ where $\varphi$ is the unique solution of (3.7) in $C_b(\mathbb{R}^{2n})$. Since the operator $K_k$ defined by

$$K_k u := Lu - \langle DU_k, D_x u\rangle, \;\; u \in D(L)$$

is dissipative in $C_b(\mathbb{R}^{2n})$ by Corollary 3.3, Eq. (3.6) is uniquely solvable in $D(L)$ for any $\lambda > 0$.

Using again Corollary 2.1, we see that $\varphi \mapsto -\langle DU_k, D_x R(\lambda, L)\varphi\rangle$ is a contraction in $C_b^1(\mathbb{R}^{2n})$ if $\lambda$ is large enough, say $\lambda > \lambda_k$. Therefore if $f \in C_b^1(\mathbb{R}^{2n})$, Eq. (3.7) has a unique solution $\varphi \in C_b^1(\mathbb{R}^{2n})$, which coincides with the unique solution in $C_b(\mathbb{R}^{2n})$, and $u_k = R(\lambda, L)\varphi$ has the claimed regularity properties by Proposition 2.1. □

Now we are ready to prove the main results of the paper.

**Theorem 3.6** *$K$ is essentially m–dissipative in $L^1(\mathbb{R}^{2n}, \mu)$. Therefore, the closure $K_1$ of $K$ in $L^1(\mathbb{R}^{2n}, \mu)$ is m–dissipative.*

*Proof* We have to show that for each $\lambda > 0$ the range of $\lambda I - K$ is dense in $L^1(\mathbb{R}^{2n}, \mu)$. Since $C_b(\mathbb{R}^{2n})$ is dense in $L^1(\mathbb{R}^{2n}, \mu)$, it is enough to show that each $f \in C_b(\mathbb{R}^{2n})$ may be approximated in $L^1(\mathbb{R}^{2n}, \mu)$ by a sequence of functions belonging to the range of $\lambda I - K$.

For each $k \in \mathbb{N}$ let $u_k \in D(L)$ be the solution of (3.6). By Proposition 3.5 we know that $D_x u_k$ is continuous and bounded, and so we can write

$$\lambda u_k - Ku_k = f + \langle DU - DU_k, D_x u_k\rangle. \tag{3.8}$$

We claim that

$$\lim_{k\to\infty} \int_{\mathbb{R}^{2n}} |\langle DU(y) - DU_k(y), D_x u_k(x, y)\rangle| \mathrm{d}\mu = 0. \tag{3.9}$$

To this aim we need an estimate on the $L^2$-norm of $|D_x u_k|$. Multiplying both sides of (3.8) by $u_k$, integrating over $\mathbb{R}^{2n}$ and taking into account (3.3) yields

$$\int_{\mathbb{R}^{2n}} u_k^2 \,\mathrm{d}\mu + \frac{1}{2}\int_{\mathbb{R}^{2n}} |D_x u_k|^2 \mathrm{d}\mu = \int_{\mathbb{R}^{2n}} f u_k \,\mathrm{d}\mu + \int_{\mathbb{R}^{2n}} \langle DU - DU_k, D_x u_k\rangle u_k \,\mathrm{d}\mu.$$

It follows that

$$\int_{\mathbb{R}^{2n}} |D_x u_k|^2 \mathrm{d}\mu \le 2\|f\|_{L^1(\mathbb{R}^{2n},\mu)}\|u_k\|_\infty + 2\int_{\mathbb{R}^{2n}} |\langle DU - DU_k, D_x u_k\rangle u_k| \mathrm{d}\mu. \tag{3.10}$$

By the Maximum Principle (Corollary 3.3),

$$\|u_k\|_\infty \le \frac{1}{\lambda}\,\|f\|_\infty. \tag{3.11}$$

Using (3.11) and then the Hölder inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} |\langle DU - DU_k, D_x u_k\rangle u_k| \mathrm{d}\mu \\
&\quad\le \frac{1}{\lambda}\,\|f\|_\infty \left(\int_{\mathbb{R}^{2n}} |DU - DU_k|^2 \mathrm{d}\mu\right)^{1/2} \left(\int_{\mathbb{R}^{2n}} |D_x u_k|^2 \mathrm{d}\mu\right)^{1/2} \\
&\quad\le \frac{1}{4}\int_{\mathbb{R}^{2n}} |D_x u_k|^2 \mathrm{d}\mu + \frac{1}{\lambda^2}\,\|f\|_\infty^2 \int_{\mathbb{R}^{2n}} |DU - DU_k|^2 \mathrm{d}\mu.
\end{aligned} \tag{3.12}$$

Consequently, by (3.10) it follows that

$$\int_{\mathbb{R}^{2n}} |D_x u_k|^2 \mathrm{d}\mu \le \frac{4}{\lambda}\,\|f\|_\infty^2 + \frac{4}{\lambda^2}\,\|f\|_\infty^2 \int_{\mathbb{R}^{2n}} |DU - DU_k|^2 \mathrm{d}\mu. \tag{3.13}$$

This yields (3.9); we have in fact

$$\begin{aligned}
&\int_{\mathbb{R}^{2n}} |\langle DU - DU_k, D_x u_k\rangle| \mathrm{d}\mu \\
&\quad\le \left(\int_{\mathbb{R}^{2n}} |DU - DU_k|^2 \mathrm{d}\mu\right)^{1/2} \left(\int_{\mathbb{R}^{2n}} |D_x u_k|^2 \mathrm{d}\mu\right)^{1/2} \\
&\quad\le \left(\int_{\mathbb{R}^{2n}} |DU - DU_k|^2 \mathrm{d}\mu\right)^{1/2} \left(\frac{4}{\lambda} + \frac{4}{\lambda^2}\int_{\mathbb{R}^{2n}} |DU - DU_k|^2 \mathrm{d}\mu\right)^{1/2} \|f\|_\infty
\end{aligned}$$

by (3.12), and the claim follows from the dominated convergence theorem.

Therefore, $K$ is essentially $m$–dissipative in $L^1(\mathbb{R}^{2n},\mu)$. By the Lumer–Phillips Theorem, its closure $K_1$ is $m$–dissipative. □

Let $T(t)$ be the strongly continuous contraction semigroup generated by $K_1$. In the following Proposition we collect some further properties of $T(t)$. We denote by $\mathbb{1} = \mathbb{1}$ the constant function with value 1.

**Proposition 3.7** *The following statements hold.*

(i) $T(t)\mathbb{1} = \mathbb{1}$, *for each* $t > 0$.
(ii) *If* $f \geq 0$ *a.e., then* $T(t)f \geq 0$ *a.e. for each* $t > 0$.
(iii) *The measure* $\mu$ *is invariant for* $T(t)$, *i.e. for each* $t > 0$ *and* $f \in L^1(\mathbb{R}^{2n}, \mu)$

$$\int_{\mathbb{R}^{2n}} T(t)f \,\mathrm{d}\mu = \int_{\mathbb{R}^{2n}} f \,\mathrm{d}\mu.$$

*Proof*

(i) Since $\mathbb{1}$ belongs to $D(L) \subset D(K_1)$ and $K_1\mathbb{1} = K\mathbb{1} = 0$, it follows that $T(t)\mathbb{1} = \mathbb{1}$, for each $t > 0$.
(ii) If $f \in C_b(\mathbb{R}^{2n})$, then $R(\lambda, K_1)f = \lim_{k\to\infty} u_k$ where the functions $u_k$ are the ones used in the proof of Theorem 3.6. Indeed, setting $f_k := \lambda u_k - Ku_k$ we have proved that $f_k$ goes to $f$ in $L^1(\mathbb{R}^{2n}, \mu)$; therefore $u_k = R(\lambda, K_1)f_k$ goes to $R(\lambda, K_1)f$ in $L^1(\mathbb{R}^{2n}, \mu)$ as $k \to \infty$. But if $f(x, y) \geq 0$ for each $(x, y)$ then $u_k(x, y) \geq 0$ for each $(x, y)$ by Lemma 3.3, and therefore $u(x, y) \geq 0$ for each $(x, y)$.
If $f \in L^1(\mathbb{R}^{2n}, \mu)$ then $R(\lambda, K_1)f = \lim_{k\to\infty} R(\lambda, K_1)f_k$, where $(f_k)$ is any sequence of functions in $C_b(\mathbb{R}^{2n})$ that converges to $f$ in $L^1(\mathbb{R}^{2n}, \mu)$. If $f(x, y) \geq 0$ a.e. we may choose a sequence $(f_k)$ such that $f_k(x, y) \geq 0$ for each $(x, y)$. Then $(R(\lambda, K_1)f_k)(x, y) \geq 0$ for each $(x, y)$ and therefore $(R(\lambda, K_1)f)(x, y) \geq 0$ for almost all $(x, y)$.
(iii) Let $f \in C_b^1(\mathbb{R}^{2n})$, fix $k \in \mathbb{N}$ and $\lambda > \lambda_k$ where $\lambda_k$ is given by Proposition 3.5, and let $u_k$ be the functions used in the proof of Theorem 3.6. By Proposition 3.5, the derivatives $D_{x_ix_j}u_k$ and $D_{x_iy_j}u_k$, $i, j = 1, \ldots n$, are continuous and bounded. We may integrate by parts and obtain

$$\int_{\mathbb{R}^{2n}} Ku_k \,\mathrm{d}\mu = 0,$$

i.e.

$$\int_{\mathbb{R}^{2n}} K_1R(\lambda, K_1)(f - \langle DU_k - DU, Du_k\rangle)\mathrm{d}\mu = 0, \quad \lambda > \lambda_k.$$

The function $\lambda \mapsto \int_{\mathbb{R}^{2n}} K_1R(\lambda, K_1)g \,\mathrm{d}\mu$ is holomorphic in the halfplane $\mathrm{Re}\,\lambda > 0$ for every $g \in L^1(\mathbb{R}^{2n}, \mu)$. Therefore,

$$\int_{\mathbb{R}^{2n}} K_1R(\lambda, K_1)(f - \langle DU_k - DU, Du_k\rangle)\mathrm{d}\mu = 0, \quad \lambda > 0,$$

and letting $k \to +\infty$

$$\int_{\mathbb{R}^{2n}} K_1R(\lambda, K_1)f \,\mathrm{d}\mu = 0, \quad \lambda > 0, \;\; f \in C_b^1(\mathbb{R}^{2n}).$$

Since $C_b^1(\mathbb{R}^{2n})$ is dense in $L^1(\mathbb{R}^{2n},\mu)$, then

$$\int_{\mathbb{R}^{2n}} K_1 R(\lambda, K_1) f \,\mathrm{d}\mu = 0, \quad \lambda > 0,\ f \in L^1(\mathbb{R}^{2n},\mu).$$

Since $D(K_1)$ is the range of $R(\lambda, K_1)$ for each $\lambda > 0$, this is equivalent to

$$\int_{\mathbb{R}^{2n}} K_1 u \,\mathrm{d}\mu = 0, \quad u \in D(K_1),$$

i.e. $\mu$ is infinitesimally invariant for $K_1$, and it implies

$$\int_{\mathbb{R}^{2n}} R(\lambda, K_1) f \,\mathrm{d}\mu = \frac{1}{\lambda}\int_{\mathbb{R}^{2n}} f \,\mathrm{d}\mu \ \ \lambda > 0, \quad f \in L^1(\mathbb{R}^{2n},\mu),$$

so that for every $k \in \mathbb{N}$

$$\int_{\mathbb{R}^{2n}} R(\lambda, K_1)^k f \,\mathrm{d}\mu = \frac{1}{\lambda^k}\int_{\mathbb{R}^{2n}} f \,\mathrm{d}\mu \ \ \lambda > 0, \quad f \in L^1(\mathbb{R}^{2n},\mu),$$

and for every $f \in L^1(\mathbb{R}^{2n},\mu)$ and $t > 0$

$$\int_{\mathbb{R}^{2n}} T(t) f \,\mathrm{d}\mu = \lim_{k\to+\infty}\int_{\mathbb{R}^{2n}} \left(\frac{k}{t}\right)^k R\left(\frac{k}{t}, K_1\right)^k f \,\mathrm{d}\mu = \int_{\mathbb{R}^{2n}} f \,\mathrm{d}\mu.$$

Statement (iii) is so proved. □

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